Covering a ball by smaller balls

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Abstract

We prove that for any covering of a unit d-dimensional Euclidean ball by smaller balls the sum of radii of the balls of the covering is greater than d. We also investigate the problem of finding necessary conditions for the sum of powers of radii of the balls covering a unit ball.

1 Introduction

Let $B \subset \mathbb{R}^d$ be a convex closed body. We say that the family of homothets $\mathcal{F} = \{\lambda_1 B, \lambda_2 B, \ldots\}$, all λ_i are from $(0,1)$, forms a translative covering of B if $B \subseteq \cup_i(\lambda_i B + x_i)$, where x_i are translation vectors in $\mathbb{R}^d$. The general question is to find necessary conditions on coefficients λ_i for existence of a translative covering. In 1990, V. Soltan formulated the following conjecture that was also stated in the book of Brass, Moser, and Pach (see [2, Conjecture 2 of Section 3.2]).

Conjecture 1 (Soltan)**.** *For any covering of a convex body $B \subset \mathbb{R}^d$ by its translative homothets with coefficients $\lambda_1, \ldots, \lambda_k$,*

$$\sum_{i=1}^{k} \lambda_i \geq d.$$

Following [5], we define

$$g(B) := \inf \left\{ \sum_{i=1}^{k} \lambda_i : B \subseteq \bigcup_{i=1}^{k} (\lambda_i B + x_i), \lambda_i \in (0,1), x_i \in \mathbb{R}^d \right\}$$

and

$$g(d) = \inf \left\{ g(B) : B \subset \mathbb{R}^d, B \text{ is a convex body} \right\}.$$

Conjecture 1 may be reformulated then as, simply, $g(d) \geq d$.

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In [8], Conjecture 1 was proven for the case $d = 2$ and all convex bodies for which there exists a covering with the sum of coefficients equal to 2 were found. In [5], the asymptotic version of the conjecture was proven, namely, it was shown that $\lim\limits_{d\to\infty} \frac{g(d)}{d} = 1$.

In this paper, we prove Conjecture 1 for the case of a d-dimensional ball $\mathbb{B}^d$ in Euclidean spaces of all dimensions d.

From the other point of view, there is an extensive literature devoted to coverings of a sphere by spherical caps. The celebrated result of Coxeter, Few, and Rogers [3] gives the lower bound of $O(d)$ for the density of coverings with sufficiently small spherical caps. Several papers such as [6, 7, 1, 9, 4] give upper bounds, typically, $O(d \log d)$, on the density of coverings or on the necessary number of caps given certain restrictions on caps' radii. From this point of view, the result of the paper is a rare exact lower bound for coverings of a sphere by spherical caps.

Theorem 1. *In any covering of the unit Euclidean sphere in $\mathbb{R}^d$, $d \geq 2$, by closed spherical caps smaller than a half-sphere, the sum of Euclidean radii of the caps is greater than d.*

Conjecture 1 for balls follows from Theorem 1 immediately. Constructing certain coverings of $\mathbb{B}^d$ we can find the value of $g(\mathbb{B}^d)$ precisely.

Corollary 1. *For $d \geq 2$, $g(\mathbb{B}^d) = d$.*

The paper is organized as follows. In Section 2, we will show how to prove Theorem 1 and Corollary 1. In Section 3, we will discuss similar problems concerning the sum of powers of radii in a covering of a ball by smaller balls.

2 Proof of the main theorem

Proof of Theorem 1. Throughout the proof, unless it is stated otherwise, by the radius of a spherical cap we always mean its Euclidean radius and by the center of a cap we mean its Euclidean center (Euclidean center of the cap's boundary). From the very beginning, we can assume that none of the caps of a covering belongs to a union of all other caps. We also note that the number of caps in a covering is always at least $d + 1$. This may be proven, for instance, by induction: for any cap, a unit subsphere of codimension 1 not intersecting it must be covered by at least d other caps.

We will prove the theorem by induction for $d \geq 2$. In the case $d = 2$, any cap of radius r (a chord with length $2r$) corresponds to a circular arc of length less than πr. Since the sum of lengths of such arcs is at least 2π, the sum of radii is greater than 2.

For the step of induction, we assume that the statement is true for $d - 1$, $d \geq 3$, and want to prove if for d.

If there are two non-intersecting spherical caps with the sum of radii at least 1, we consider a unit subsphere of codimension 1 separating them. By the induction hypothesis, the sum of radii of spherical caps covering it must be greater than $d - 1$ and, in total, the sum of radii of all caps is greater than d. From this moment on we assume there are no pairs of caps like this.

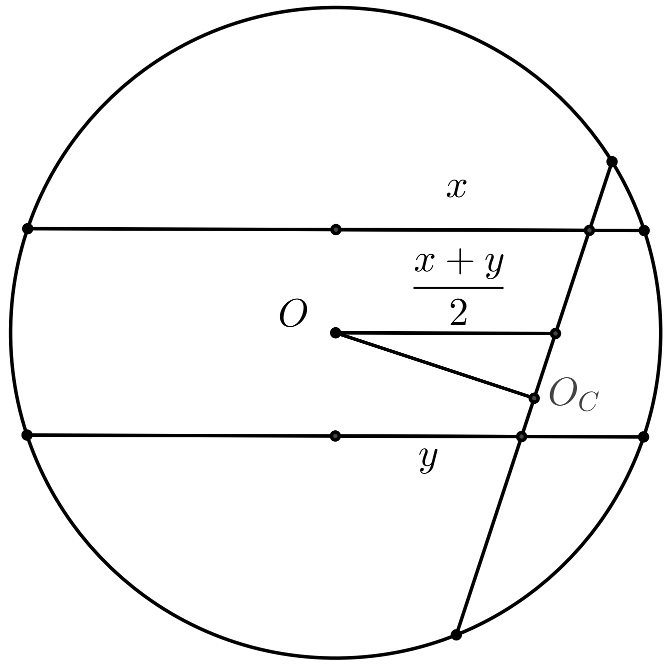

Figure 1: $\frac{x+y}{2} \geq OO_C$.

If, on the other hand, there are two intersecting spherical caps with the sum of radii less than 1, we can substitute them by a bigger spherical cap covering both of them with a radius less than the sum of their radii. If the statement of the theorem is true after the substitution, then it was true before the substitution as well. By making substitutions of this kind, we can guarantee there are no pairs of caps like this either.

We consider any maximal spherical cap C_{max} of a given covering $\mathcal{C}$ of a unit sphere $\mathbb{S}^{d-1}$. Denote the boundary of this spherical cap by S and denote its radius by R.

By S' we denote the sphere centrally symmetric to S with respect to the center of $\mathbb{S}^{d-1}$. We claim that it is sufficient to consider only the situation when all other spherical caps of the covering intersect both S and S'. Consider an arbitrary cap C from $\mathcal{C}$. Since C intersects some other cap, the sum of the radii of C and this cap must be at least 1. Hence the sum of the radii of C and C_{max} must be at least 1 and they must intersect. If C does not intersect S', then, analogously to the case shown above, we can substitute C_{max} and C by a cap covering both of them. Due to the triangle inequality, the radius of the substituting cap will be smaller that the sum of the radii of C_{max} and C. By making such substitutions we can obtain a covering where the required condition holds.

Now assume that the radius of the cap C intersecting S and S' is r. Denote the distance from the center of S to the center of $S \cap C$ by x and from the center of S' to the center of $S' \cap C$ by y. Then the distance from the center of $\mathbb{S}^{d-1}$ to the center of C is not greater than $\frac{x+y}{2}$ (see Figure 1 with the orthogonal projection along $S \cap C$). Hence $(\frac{x+y}{2})^2 \geq 1 - r^2$. From this inequality and Jensen's inequality for the concave function $f(t) = \sqrt{R^2 - t^2}$, we get

$$\sqrt{R^2-x^2}+\sqrt{R^2-y^2}\le 2\sqrt{R^2-\left(\frac{x+y}{2}\right)^2}\le 2\sqrt{R^2+r^2-1}. \tag{1}$$

We note that the left hand side of this inequality contains the sum of radii of $S\cap C$ and $S'\cap C$.

Assume we have k caps C_1, …, C_k in the covering, not including C_{max}, and define r_i, x_i, y_i for them just as above. Summing up the inequalities (1) for all these caps, we get

$$\sum_{i=1}^{k}\left(\sqrt{R^2-x_i^2}+\sqrt{R^2-y_i^2}\right)\le 2\sum_{i=1}^{k}\sqrt{R^2+r_i^2-1}.$$

The left hand side of this inequality is the sum of radii of the coverings of S and S' and, by the induction hypothesis, must be greater than $2(d-1)R$. Therefore,

$$(d-1)R<\sum_{i=1}^{k}\sqrt{R^2+r_i^2-1}. \tag{2}$$

Using Jensen's inequality for the concave function $g(t)=\sqrt{t^2-(1-R^2)}$ and the fact that $k\ge d$,

$$\begin{aligned}\sum_{i=1}^{k}\sqrt{R^2+r_i^2-1}\le k\sqrt{\left(\frac{\sum_{i=1}^{k}r_i}{k}\right)^2+R^2-1}=\\ \sqrt{\left(\sum_{i=1}^{k}r_i\right)^2-k^2(1-R^2)}\le\sqrt{\left(\sum_{i=1}^{k}r_i\right)^2-d^2(1-R^2)}.\end{aligned} \tag{3}$$

Combining inequalities (2) and (3), we get

$$\sqrt{\left(\sum_{i=1}^{k}r_i\right)^2-d^2(1-R^2)}>(d-1)R,\ \text{ so }\sum_{i=1}^{k}r_i>\sqrt{(d-1)^2R^2+d^2(1-R^2)}.$$

Therefore, the sum of radii of all caps in the covering satisfies the inequality

$$R+\sum_{i=1}^{k}r_i\ >R+\sqrt{(d-1)^2R^2+d^2(1-R^2)},$$

which is at least d for any $R\in[0,1]$. □

Proof of Corollary 1. Theorem 1 implies that $g(\mathbb{B}^d)\ge d$. In order to prove the equality, it is sufficient to show that, for any given $\varepsilon>0$, there is a set of balls covering the unit ball with the sum of radii less than $d+\varepsilon$.

Fix a positive $\delta < \frac{1}{d}$. In a $(d-1)$-dimensional subspace (all points with the last coordinate 0) consider a sphere S_δ with center at the origin and radius δ. We choose points $v_1, \ldots, v_d$ on S_δ so that they form a regular $(d-1)$-dimensional simplex. We also take points $v_+ = (0, \ldots, 0, \sqrt{1-d^2\delta^2})$ and $v_- = -v_+$. We claim that the set consisting of d balls with centers at $v_1, \ldots, v_d$ and radius $1 - \frac{1}{2}\delta^2$ and two balls with centers at v_+, v_- and radius $d\delta$ covers the d-dimensional unit ball with the center at the origin.

Consider an arbitrary point u such that one of the angles $\angle(uv_i0)$ is obtuse. Then in the case u does not belong to a ball with the center v_i, we have $u0^2 > uv_i^2 + v_i0^2 > (1-\frac{1}{2}\delta^2)^2 + \delta^2 > 1$.

For each i, $\angle(uv_i0)$ is not obtuse if u belongs to a half-space formed by a hyperplane through v_i and perpendicular to $0v_i$. The intersection of these half-spaces is an infinite cylinder. The last coordinate axis is the axis of this cylinder. The base of the cylinder is formed by the regular simplex dual to the simplex formed by all v_i with respect to S_δ. The distance from the vertices of the base to the origin is $d\delta$. Hence the intersections of the cylinder at the base vertices with the unit sphere will be distant from the axis of the last coordinate by $d\delta$ and will have the last coordinate of $\pm\sqrt{1-d^2\delta^2}$ which will be covered by the balls with the centers v_+ and v_-.

Points $(0, \ldots, 0, \pm 1)$ are also covered by the balls with the centers v_+ and v_- because $1 - \sqrt{1-d^2\delta^2} < d\delta$.

The spheres with the centers $v_1, \ldots, v_d$ have two common points on the last coordinate axis with the coordinates $\pm\sqrt{(1-\frac{1}{2}\delta^2)^2 - \delta^2}$. Since $\sqrt{(1-\frac{1}{2}\delta^2)^2 - \delta^2} > \sqrt{1-2\delta^2} > \sqrt{1-d^2\delta^2}$, these points also belong to the balls with the centers v_+ and v_-.

All other points are covered due to the convexity of balls.

The sum of radii of the balls in the family is $d(1-\frac{1}{2}\delta^2) + 2d\delta$, which is less than $d + \varepsilon$ for sufficiently small δ. $\square$

3 Sum of powers of radii

Similarly to $g(B)$ and $g(d)$, we can define $g_\alpha(B)$ and $g_\alpha(d)$ for the sums of powers of the homothety coefficients:

$$g_\alpha(B) := \inf\left\{\sum_{i=1}^{k} \lambda_i^\alpha : B \subseteq \bigcup_{i=1}^{k}(\lambda_i B + x_i), \lambda_i \in (0,1), x_i \in \mathbb{R}^d\right\},$$

$$g_\alpha(d) = \inf\left\{g_\alpha(B) : B \subset \mathbb{R}^d, B \text{ is a convex body}\right\}.$$

The result from [5] may be formulated in these terms as $\lim\limits_{d\to\infty} \frac{g_\alpha(d)}{d} = 1$ for any fixed $\alpha \geq 1$.

We will show that results somewhat similar to Theorem 1 and Corollary 1 hold for certain other values of α as well.

Proposition 1. *1. For any $\alpha > d$ and any natural d, $g_\alpha(\mathbb{B}^d) = 0$.*

2. *For any $\alpha \in (d-1, d]$ and any natural d, $g_\alpha(\mathbb{B}^d) = 1$.*

3. *For any $\alpha \in (d-2, d-1]$ and any natural $d \geq 2$, $g_\alpha(\mathbb{B}^d) = 2$.*

Proof. To prove this proposition we use the following observations about coverings of a unit ball by balls with radii $\lambda_1, \ldots, \lambda_k$: $\sum_{i=1}^{k} \lambda_i^d \geq 1$ because the whole volume of the ball is covered and $\sum_{i=1}^{k} \lambda_i^{d-1} \geq 2$ because the whole spherical surface area is covered by smaller half-spheres. From these observations we get lower bounds for all cases of the proposition. It remains to construct coverings with sufficiently close sums of the powers of radii.

For the first case, it is sufficient to take any covering of $\mathbb{R}^d$ by equal balls with sufficiently small radii and the density of the covering $O(d \log d)$ (see [6]).

For the second case, we consider the covering of the unit sphere by spherical caps of sufficiently small size and the density of the covering $O(d \log d)$ (see [1, 9, 4]) and add one ball with radius sufficiently close to 1 and concentric to the unit ball.

For the third case, we can take two congruent balls with coordinates of centers $(0, \ldots, 0, \pm\Delta)$ with sufficiently small Δ and radius sufficiently close to 1. The remaining part is a neighborhood of a spherical $(d-2)$-dimensional equator which we can cover by sufficiently small equal spheres with the density of the covering $O(d \log d)$. $\square$

Interestingly, for all cases we resolved so far $g_\alpha(\mathbb{R}^d) = d + 1 - \lceil \alpha \rceil$ (unless $\alpha > d + 1$, when the value of g is 0). This motivates us to formulate the following conjecture.

Conjecture 2. *For all natural d and all α such that $0 \leq \alpha \leq d+1$, $g_\alpha(\mathbb{B}^d) = d + 1 - \lceil \alpha \rceil$.*

Combining our results, we conclude that Conjecture 2 is true when $\alpha \in [0, 1] \cap (d-2, d+1]$ for any $d \geq 2$.

Although we don't know how to prove the corresponding lower bounds for $g_\alpha(\mathbb{B}^d)$, it is possible to confirm upper bounds from Conjecture 2.

Theorem 2. *For all natural d and all α such that $0 \leq \alpha \leq d+1$, $g_\alpha(\mathbb{B}^d) \leq d + 1 - \lceil \alpha \rceil$.*

Proof. Assume $\alpha \in (n, n+1]$, where $n \in \mathbb{N}$. We will use the construction generalizing both constructions from Corollary 1 and from Proposition 1. We take a $(d-n-1)$-dimensional space Π containing the center 0 of the unit ball $\mathbb{B}^d$ and consider a sphere S_δ in this space with center at 0 and radius $\delta > 0$. We select $d - n$ points from S_δ forming a regular simplex. Now for our covering we choose equal balls with centers at these points and radius $1 - \frac{1}{2}\delta^2$. By $S_\perp$ we denote an intersection of the orthogonal complement $\Pi^\perp$ and the boundary of $\mathbb{B}^d$. The set of points not covered by the balls we have already chosen is an $O(\delta)$-neighborhood of $S^\perp$. Since $S^\perp$ belongs to the n-dimensional space $\Pi^\perp$, we can choose its covering with sufficiently small spheres with density $O(n \log n)$ and obtain a sufficiently small contribution of this covering into the sum of powers of radii. Overall, we will get the sum of powers of radii as close to $d - n$ as we wish. $\square$

It is worth mentioning that, due to Theorem 2, in order to prove Conjecture 2 it is sufficient to prove lower bounds for natural α.

Although we don't know how to prove Conjecture 2, we can find a universal lower bound for $g_\alpha(\mathbb{B}^d)$.

Theorem 3. *For $d \geq 3$ and any $\alpha \in [0, d]$ $g_\alpha(\mathbb{B}^d) \geq d - \alpha \ln^2 d$.*

Proof. We will prove this theorem by induction for d. Conjecture 2 is true $d = 3$ so we can use it as the base of the induction. For α such that $d - \alpha \ln^2 d \leq 1$, the statement of the theorem holds immediately so we assume that $\alpha < \frac{d-1}{\ln^2 d}$.

The radii of a covering $\lambda_1, \ldots, \lambda_k$ satisfy, as we mentioned above, $\sum\limits_{i=1}^{k} \lambda_i^{d-1} \geq 2$. Assume λ_1 is the largest radius of the covering balls. Then

$$\sum_{i=1}^{k} \lambda_i^\alpha \geq \frac{\sum\limits_{i=1}^{k} \lambda_i^{d-1}}{\lambda_1^{d-\alpha-1}} \geq \frac{2}{\lambda_1^{d-\alpha-1}}.$$

If $\lambda_1^{d-\alpha-1} \leq \frac{2}{d-\alpha\ln^2 d}$ holds, the statement of the theorem is true. Hence it is sufficient to consider the case when $\lambda_1^{d-\alpha-1} > \frac{2}{d-\alpha\ln^2 d}$

We consider an arbitrary unit subsphere of codimension 1 non-intersecting the largest ball of the covering. By the induction hypothesis, the sum of the α-powers of the radii of $(d-1)$-dimensional balls covering it should be at least $(d-1) - \alpha \ln^2(d-1)$. Hence we get

$$\sum_{i=1}^{k} \lambda_i^\alpha \geq \lambda_1^\alpha + (d-1) - \alpha \ln^2(d-1) > \left(\frac{2}{d}\right)^{\frac{\alpha}{d-\alpha-1}} + (d-1) - \alpha \ln^2(d-1).$$

It remains to show that

$$\left(\frac{2}{d}\right)^{\frac{\alpha}{d-\alpha-1}} + (d-1) - \alpha \ln^2(d-1) \geq d - \alpha \ln^2 d;$$

$$\left(\frac{2}{d}\right)^{\frac{\alpha}{d-\alpha-1}} + \alpha(\ln^2 d - \ln^2(d-1)) \geq 1.$$

For $d \geq 4$, $\ln^2 d - \ln^2(d-1) \geq \frac{2\ln d}{d}$. Hence it is sufficient to prove

$$\left(\frac{2}{d}\right)^{\frac{\alpha}{d-\alpha-1}} \geq 1 - \frac{2\alpha \ln d}{d}.$$

Using $1 - \frac{2\alpha\ln d}{d} \leq e^{-\frac{2\alpha\ln d}{d}} = \frac{1}{d^{2\alpha/d}}$, we are left with

$$\left(\frac{2}{d}\right)^{\frac{1}{d-\alpha-1}} \geq \left(\frac{1}{d}\right)^{\frac{2}{d}};$$

$$d^{2\frac{d-\alpha-1}{d}} \geq \frac{d}{2},$$

which is true for any $d \geq 4$ and any $\alpha < \frac{d-1}{\ln^2 d}$. □

We note that, in the case of Euclidean balls, this result generalizes the result of Naszódi [5] since, for a fixed α, $\lim\limits_{d\to\infty} \frac{d-\alpha\ln^2 d}{d} = 1$, and also covers certain cases when α depends on d, for instance, $\alpha \sim d^c$ for all c from $(0, 1)$.

Concluding the paper, we would like to make the general conjecture that the same lower bounds as in Conjecture 2 hold for all convex bodies.

Conjecture 3. *For all natural d and all α such that $0 \leq \alpha \leq d+1$, $g_\alpha(d) = d + 1 - \lceil\alpha\rceil$.*

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